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The inertial lift on a rigid sphere in a linear shear flow field near a flat wall†

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An expression which predicts the inertial lift, to lowest order, on a rigid sphere translating in a linear shear flow field near a flat infinite wall has been derived. This expression may be used when the wall lies within the inner region of the sphere's disturbance flow. It is valid even when the gap is small compared to the radius of the sphere. When the sphere is far from the wall, the lift force predicted by the present analysis converges to the value predicted by earlier analyses which consider the sphere as a point force or a force doublet singularity. The effect of rotation of the sphere on the lift has also been analysed.

1. Introduction

A rigid sphere that translates parallel to a flat wall, in a fluid undergoing a uni-directional shear flow parallel to the wall, experiences a force in a direction normal to the wall. This transverse force is due to finite inertial effects. Since the Stokes equation and the boundary conditions are linear, a reversibility argument can be used to show that the Stokes solution does not predict any lift force (see Pozrikidis 1992). The effect of small but finite inertial terms in the Navier–Stokes equation has been studied by perturbation methods. Saffman (1965) analysed the problem of a rigid sphere translating in a direction parallel to the streamlines of a uni-directional unbounded linear shear flow field. Saffman applied the method of matched asymptotic expansions and obtained an expression for the lift force. Saffman's expression was derived by considering the linearized ('Oseen like') equations of motion in the outer region and treating the sphere as a point force singularity to leading order. He assumed that the inertial terms due to shear are large compared to the inertial terms due to the slip velocity and that all the Reynolds numbers for the flow are small compared to unity. The Saffman lift arises due to inertial effects at distances that are $O((\nu/G)^{\frac{1}{2}})$; i.e. the outer flow field (here G denotes the velocity gradient and ν denotes the kinematic viscosity of the fluid). Saffman also showed that the effect of rotation of the sphere is to produce a lift force that acts in the same direction as the lift due to shear and that this contribution due to the rotation is a higher-order effect. The lift force due to rotation was shown to be equal to that given by Rubinow & Keller's (1961) expression for the lift on a sphere translating and spinning in a quiescent unbounded fluid.

Cox & Brenner (1968) analysed the case of a particle in a flow field bounded by a system of walls. In general, at low Reynolds numbers, asymptotic series for velocity and pressure fields are not uniformly valid, and different asymptotic representations for an inner region and an outer region that match in an appropriate overlap domain

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have to be used. However, Cox & Brenner showed that, when the walls lie in the inner region of the asymptotic expansion (i.e. $l \ll \nu/V$, where l and V are the distance between the centre of the sphere and the wall and a characteristic velocity, respectively), for certain flows, the first-order field satisfies homogeneous boundary conditions on the particle and at infinity. They showed that the lift force, to lowest order, could be evaluated without considering the flow in the outer region. These authors obtained general expressions for the lift force in terms of Green's functions by assuming that the sphere is very far from the wall and that the disturbance flow due to the sphere near the wall can be considered to be due to a Stokeslet (or a force doublet if the sphere is neutrally buoyant). These general expressions for the lift force can be applied if the Reynolds number is small and the walls lie in the inner region and the distance between the wall and the centre of the sphere is large compared to the radius of the sphere.

Ho & Leal (1974) considered the lift on a neutrally buoyant, freely rotating sphere in a planar flow bounded by two walls and derived an expression for the lift force by the method of reflections. This expression is valid when the sphere is not very close to the walls.

Cox & Hsu (1977) showed that the class of wall-bounded flows for which the lift force to the lowest order could be computed without considering an outer expansion includes wall-bounded linear shear flows and quadratically varying flows. These authors evaluated the Green's functions in Cox & Brenner's (1968) expressions and evaluated the lift force for a sphere sedimenting near a flat wall in a stagnant fluid and neutrally buoyant and non-neutrally buoyant spheres in a fluid undergoing a planar quadratically varying flow. Cox & Hsu's (1977) expression can be used when the distance from the wall is large compared to the radius of the sphere. Vasseur & Cox (1976) extended this analysis to the case of flows bounded by two infinitely large flat parallel walls.

Leighton & Acrivos (1985) evaluated the lift on a stationary sphere in a shear flow when the sphere touches the wall. They evaluated the lift to lowest order in Reynolds number. According to their analysis, the lift points away from the wall and varies as the fourth power of the radius of the sphere and the square of the velocity gradient.

Drew (1988) applied perturbation techniques to evaluate the lift on a sphere translating in a shear field past a distant wall. The sphere was assumed to be very far from the wall and treated as a point force and it was also assumed that the inertial terms due to shear are large compared to the inertial terms due to the slip velocity of the sphere. The lift force was evaluated by solving an ordinary differential equation for the Fourier transform of the velocity field.

Schonberg & Hinch (1989) analysed the lift on a neutrally buoyant sphere in a plane Poiseuille flow. The sphere was treated as a force dipole singularity and singular perturbation techniques were used to evaluate the inertial migration velocity of the sphere. A system of coupled differential equations with the multipole singularities replaced by jump conditions across the location of the particle were solved numerically to obtain the migration velocity in the Fourier space. The migration velocity was subsequently evaluated by numerical inversion of the Fourier transform.

McLaughlin (1991) generalized Saffman's (1965) analysis of unbounded shear flows to the case when the inertial effects due to slip are comparable with the inertial effects due to shear and derived an expression for the inertial lift on a sphere in an unbounded shear flow field. The only assumption made was that the Reynolds numbers are small compared to unity. According to this analysis, when the inertial effects due to the slip velocity are larger than the inertial effects due to shear, the lift force is much smaller than that predicted by Saffman's (1965) expression.

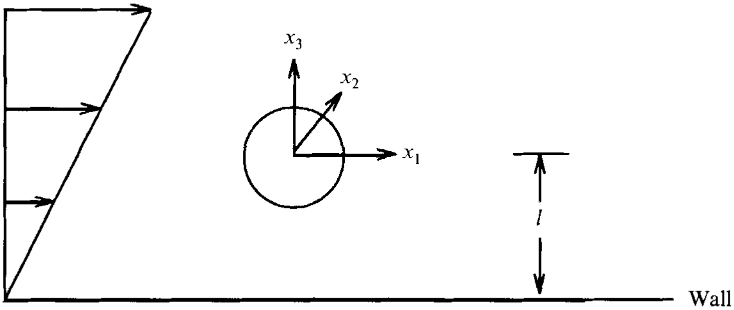

FIGURE 1. The coordinate system used to describe the motion of a rigid sphere near a flat wall.

The inertial lift on a sphere translating in a shear flow bounded by a single flat infinite wall was analysed by McLaughlin (1993). He derived an expression for the lift force by superposition of the disturbance flow created by the wall and the migration velocity due to an unbounded shear field. This analysis is applicable when the wall lies in the outer region or the inner region provided the distance between the wall and the sphere is large compared to the radius of the sphere. It was shown that the lift force converges asymptotically to the value predicted by Cox & Hsu's (1977) expression when the distance from the wall decreases. This analysis has been extended to the case in which the sphere translates in a linear shear flow between two parallel flat walls by Cherukat, McLaughlin & Graham (1994).

All the analyses for wall-bounded shear flows in which the sphere does not touch the wall assume that the distance between the sphere and the wall is large compared to the radius of the sphere and treat the particle as a point force or a force doublet. There are several situations of practical interest in which it is required to know the lift on a spherical particle when it is very close to a flat wall. In this paper we will consider the inertial lift on a rigid sphere translating in a shear flow near a flat wall when the distance between the sphere and the wall and the radius of the sphere are comparable. The lift force on rotating and non-rotating spheres will be analysed.

The study of inertial lift at low Reynolds numbers has several practical applications. In the deposition of aerosol particles on a solid surface from a turbulent gas stream, it has been found (McLaughlin 1989) that the lift force is important in determining the trajectory of the particles in the viscous sublayer near the solid boundaries. The inertial lift is also important in fluid flow fractionation of particles in a suspension (see Johansson, Olgard & Jerqvist 1970).

2. Asymptotic analysis

Consider a rigid sphere of radius a near a flat wall in a Newtonian incompressible fluid of dynamic viscosity μ and density ρ. The distance between the wall and the centre of the sphere is l. The parameter κ is defined as

$$\kappa = a/l. \tag{2.1}$$

In the absence of the sphere the fluid undergoes a uni-directional linear shear flow. The velocity gradient of this flow field in the direction normal to the wall is G, and G is assumed to be positive. The sphere moves with a velocity V_{sph} parallel to the wall and rotates with an angular speed ω about an axis parallel to the wall and normal to the direction of translation. The slip velocity, U_s, is defined by the equation

$$V_{sph} = Gl + U_s. \tag{2.2}$$

In a reference frame (x_1', x_2', x_3') with its origin at the centre of the sphere and that moves with the sphere (see figure 1), the fluid at an infinite distance from the sphere has a velocity $(G(l+x_3')-V_{sph})\delta_{i1}$; where δ_{ij} denotes the Kronecker delta. If V is a characteristic velocity and a, the sphere radius, is chosen the characteristic length, a Reynolds number Re can be defined as

$$Re = aV/\nu. \tag{2.3}$$

The distances and the velocities are made non-dimensional by dividing by a and V respectively. The Navier–Stokes equation, the continuity equation and the associated boundary conditions can be written in dimensionless form as

$$Re\left(u_j \frac{\partial u_i}{\partial x_j}\right) = \frac{\partial \sigma_{ij}}{\partial x_j}, \tag{2.4}$$

$$\frac{\partial u_j}{\partial x_j} = 0, \tag{2.5}$$

$$u_i = \epsilon_{ijk}\left(\frac{\omega a}{V}\right)\delta_{j2}x_k \quad \text{when} \quad r = 1, \tag{2.6}$$

$$u_i = -\left(\frac{V_{sph}}{V}\right)\delta_{i1} \quad \text{when} \quad x_3 = -(l/a), \tag{2.7}$$

$$u_i = \left[\frac{Ga}{V}\left(\frac{l}{a}+x_3\right)-\frac{V_{sph}}{V}\right]\delta_{i1} \quad \text{as} \quad r\rightarrow\infty, \tag{2.8}$$

where σ_{ij} is the dimensionless stress tensor and r is the magnitude of the position vector. It is assumed that $|Re| \ll 1$ and that the velocity and pressure fields can be expressed as asymptotic power series in Re; i.e.

$$u_i = u_i^{(0)} + Re\, u_i^{(1)} + o(Re), \tag{2.9}$$

and

$$p = p^{(0)} + Re\, p^{(1)} + o(Re). \tag{2.10}$$

The stress tensor associated with the field $(u_i^{(0)}, p^{(0)})$ is $\sigma_{ij}^{(0)}$ and the stress tensor associated with the field $(u_i^{(1)}, p^{(1)})$ is $\sigma_{ij}^{(1)}$.

The zeroth-order field (creeping flow solution) is a uniformly valid approximation and satisfies the following equations and boundary conditions:

$$\frac{\partial \sigma_{ij}^{(0)}}{\partial x_j} = 0, \tag{2.11}$$

$$\frac{\partial u_j^{(0)}}{\partial x_j} = 0, \tag{2.12}$$

$$u_i^{(0)} = \epsilon_{ijk}\left(\frac{\omega a}{V}\right)\delta_{j2}x_k \quad \text{when} \quad r = 1, \tag{2.13}$$

$$u_i^{(0)} = -\left(\frac{V_{sph}}{V}\right)\delta_{i1} \quad \text{when} \quad x_3 = -(l/a), \tag{2.14}$$

$$u_i^{(0)} = \left[\frac{Ga}{V}\left(\frac{l}{a}+x_3\right)-\frac{V_{sph}}{V}\right]\delta_{i1} \quad \text{as} \quad r\rightarrow\infty. \tag{2.15}$$

The lift force due to the creeping flow field is zero. Hence, if $\sigma_{ij}^{(0)}$ is integrated over the surface of the sphere, the x_3 component of the force will be zero. To compute the lift to lowest order, it is necessary to integrate the first-order stress tensor $\sigma_{ij}^{(1)}$ over the surface of the sphere and determine the x_3 component of the force. The first-order field satisfies the equations

$$\frac{\partial \sigma_{ij}^{(1)}}{\partial x_j} = u_j^{(0)} \frac{\partial u_i^{(0)}}{\partial x_j}, \tag{2.16}$$

$$\frac{\partial u_j^{(1)}}{\partial x_j} = 0, \tag{2.17}$$

and the boundary conditions

$$u_i^{(1)} = 0 \quad \text{when} \quad r = 1, \tag{2.18}$$

$$u_i^{(1)} = 0 \quad \text{when} \quad x_3 = -(l/a). \tag{2.19}$$

The boundary condition (2.19) implies that the wall lies in the inner region. The inner region is that region in which the viscous effects dominate and the effect of inertia can be considered to be a small perturbation. In the outer region, inertial effects and viscous effects are comparable and the equations of motion can be approximated to leading order by an 'Oseen like' equation (see Proudman & Pearson 1957 and Saffman 1965). Hence the first-order inner expansion does not satisfy the boundary conditions at infinity. However, it can be made to satisfy the boundary condition on the sphere. It can also be made to satisfy the boundary condition on the wall if the distance between the sphere and the wall is small compared to the lengthscale at which the inertial effects become comparable with the viscous effects. For a sphere translating with a velocity U_s in a quiescent fluid, inertial effects become comparable with viscous effects at distances which are $O(L_s)$, where L_s is the Stokes length defined by

$$L_s = \nu / U_s. \tag{2.20}$$

If the sphere is translating in a strong shear field, inertial effects and viscous effects are comparable at distances which are of $O(L_G)$, where L_G is the Saffman length defined by

$$L_G = (\nu/G)^{\frac{1}{2}}. \tag{2.21}$$

These lengthscales can be derived by formally scaling the equations of motion to make the viscous terms and the inertial terms of the same order (see Proudman & Pearson 1957 and Saffman 1965). The wall will lie in the inner region if the inequality

$$l \ll \min(L_s, L_G) \tag{2.22}$$

is satisfied.

In general, the first-order velocity field will not satisfy homogeneous boundary conditions at infinity and has to be determined by matching to the outer flow field. However, for a sphere translating in a stagnant fluid near the wall, it can be shown that the first-order velocity field satisfies homogeneous boundary conditions at infinity (Cox & Brenner 1968). Though this is not true when the shear is non-zero, Cox & Hsu (1977) have shown that the force on the sphere to lowest order in *Re* can be determined without considering the outer flow field. Cox & Brenner (1968) assumed that $\kappa \ll 1$ and approximated the disturbance flow near the wall due to the sphere as due to a point force and a force doublet and derived an expression for the lift force in terms of Green's functions. In this analysis we will consider the case for which the parameter κ is $O(1)$. Hence, the sphere cannot be treated as a point singularity.

The dimensionless lift force on the sphere, F_l, is given by

$$F_l = Re \iint \sigma_{ij}^{(1)} n_j \delta_{i3} \, \mathrm{d}S, \tag{2.23}$$

where the surface integral is evaluated over the surface of the sphere. It can be shown using the generalized reciprocal theorem that the surface integral in equation (2.23) can be expressed as a volume integral involving the creeping flow solution $u_i^{(0)}$ and another creeping flow solution $v_i^{(0)}$, where $v_i^{(0)}$ is the creeping flow solution when the sphere translates towards the wall with unit velocity in a quiescent fluid (see Cox 1965; Cox & Brenner 1968; Ho & Leal 1974; Leighton & Acrivos 1985). Using this technique the dimensionless lift force can be expressed as

$$F_l = Re \iiint v_i^{(0)} u_j^{(0)} \frac{\partial u_i^{(0)}}{\partial x_j} \mathrm{d}V, \tag{2.24}$$

where the integral is evaluated over the entire space occupied by the fluid. It can also be shown that this integral is convergent (see Leighton & Acrivos 1985). Thus, the lift force to lowest order can be computed if the fields $u_i^{(0)}$ and $v_i^{(0)}$ are known. The dimensional lift force, F_l', can be expressed as

$$F_l' = a\mu V Re \iiint v_i^{(0)} u_j^{(0)} \frac{\partial u_i^{(0)}}{\partial x_j} \mathrm{d}V. \tag{2.25}$$

2.1. *Computation of the lift force*

The creeping flow solution for a rigid sphere in a three-dimensional shear flow field moving in the presence of an arbitrarily located second rigid sphere has been derived by Lin, Lee & Sather (1970). This general solution can be used to evaluate the flow fields $u_i^{(0)}$ and $v_i^{(0)}$. The solution is given in terms of a spherical bi-polar coordinate system (ξ, η, ϕ) and an associated cylindrical polar coordinate system (ρ, z, ϕ). The bi-polar coordinate system is defined by

$$\rho = \frac{c \sin \eta}{\cosh \xi - \cos \eta}, \tag{2.26}$$

$$z = \frac{c \sinh \xi}{\cosh \xi - \cos \eta}, \tag{2.27}$$

$$\alpha = \cosh^{-1}(l/a), \tag{2.28}$$

$$c = \sinh \alpha. \tag{2.29}$$

The coordinate surface $\xi = \alpha$ corresponds to the sphere (the centre of the sphere being located at $z = l$, $\rho = 0$) and the coordinate surface $\xi = 0$ corresponds to a sphere of infinite radius which coincides with the wall. The cylindrical polar components (u_ρ, u_ϕ, u_z) of the velocity field $u_i^{(0)}$ are given in the following functional form:

$$u_\rho = \bar{U}_\rho(\rho, z, \Lambda_G, \Lambda_\omega, \Lambda_s) \cos \phi, \tag{2.30}$$

$$u_\phi = \bar{U}_\phi(\rho, z, \Lambda_G, \Lambda_\omega, \Lambda_s) \sin \phi, \tag{2.31}$$

$$u_z = \bar{U}_z(\rho, z, \Lambda_G, \Lambda_\omega, \Lambda_s) \cos \phi, \tag{2.32}$$

where $$\Lambda_G = Ga/V, \quad \Lambda_\omega = \omega a/V, \quad \text{and} \quad \Lambda_s = V_{sph}/V. \tag{2.33}$$

If the Reynolds numbers Re_G and Re_ω are defined as

$$Re_G = Ga^2/\nu \tag{2.34}$$

and $$Re_\omega = \omega a^2/\nu \tag{2.35}$$

then $$\Lambda_G = Re_G/Re, \quad \text{and} \quad \Lambda_\omega = Re_\omega/Re. \tag{2.36}$$

The functions $\bar{U}_\rho$, $\bar{U}_\phi$ and $\bar{U}_z$ depend on Λ_G, Λ_ω and Λ_s and are infinite series in the coordinates ξ and η. The coefficients of the terms in these series depend only on the value of l/a. The number of terms that have to be retained in these series is not known *a priori*. The coefficients of the terms in the series can be determined by truncating the series after a certain number of terms and computing the coefficients. To ensure that a sufficient number of terms have been retained, the number of terms in the series should be increased and the magnitudes of the coefficients should be checked to see if they are sufficiently small and can be neglected. The number of terms that have to be retained in the series to achieve sufficient accuracy increases as l/a becomes small.

The integral in (2.24) can be expressed in cylindrical polar coordinates and the dimensionless lift force can be expressed as

$$F_l = Re\int_0^{2\pi}\int_0^{\alpha}\int_0^{\pi}\left[v_\rho^{(0)}\left(u_\rho\frac{\partial u_\rho}{\partial\rho}+\frac{u_\phi}{\rho}\frac{\partial u_\rho}{\partial\phi}-\frac{u_\phi^2}{\rho}+u_z\frac{\partial u_\rho}{\partial z}\right)\right.$$
$$\left.+v_z^{(0)}\left(u_\rho\frac{\partial u_z}{\partial\rho}+\frac{u_\phi}{\rho}\frac{\partial u_z}{\partial\phi}+u_z\frac{\partial u_z}{\partial z}\right)\right]\frac{c^3\sin\eta}{(\cosh\xi-\cos\eta)^3}\,\mathrm{d}\eta\,\mathrm{d}\xi\,\mathrm{d}\phi. \quad (2.37)$$

The expressions for the cylindrical polar components of the velocity fields can be substituted in (2.37) and the integration in the ϕ-coordinate can be done analytically. Thus the expression for the dimensionless lift force reduces to

$$F_l = Re\,\pi\int_0^{\alpha}\int_0^{\pi}\left[v_\rho^{(0)}\left(\bar{U}_\rho\frac{\partial\bar{U}_\rho}{\partial\rho}-\frac{\bar{U}_\phi\bar{U}_\rho}{\rho}-\frac{\bar{U}_\phi^2}{\rho}+\bar{U}_z\frac{\partial\bar{U}_\rho}{\partial z}\right)\right.$$
$$\left.+v_z^{(0)}\left(\bar{U}_\rho\frac{\partial\bar{U}_z}{\partial\rho}-\frac{\bar{U}_\phi\bar{U}_z}{\rho}+\bar{U}_z\frac{\partial\bar{U}_z}{\partial z}\right)\right]\frac{c^3\sin\eta}{(\cosh\xi-\cos\eta)^3}\,\mathrm{d}\eta\,\mathrm{d}\xi. \quad (2.38)$$

Equation (2.38) can be written compactly as

$$F_l = Re\,I, \quad (2.39)$$

where I is the integral defined by

$$I = \pi\int_0^{\alpha}\int_0^{\pi}\left[v_\rho^{(0)}\left(\bar{U}_\rho\frac{\partial\bar{U}_\rho}{\partial\rho}-\frac{\bar{U}_\phi\bar{U}_\rho}{\rho}-\frac{\bar{U}_\phi^2}{\rho}+\bar{U}_z\frac{\partial\bar{U}_\rho}{\partial z}\right)\right.$$
$$\left.+v_z^{(0)}\left(\bar{U}_\rho\frac{\partial\bar{U}_z}{\partial\rho}-\frac{\bar{U}_\phi\bar{U}_z}{\rho}+\bar{U}_z\frac{\partial\bar{U}_z}{\partial z}\right)\right]\frac{c^3\sin\eta}{(\cosh\xi-\cos\eta)^3}\,\mathrm{d}\eta\,\mathrm{d}\xi. \quad (2.40)$$

The integral, I, has to be evaluated numerically to obtain the lift force.

3. Discussion

The dimensionless lift force was computed by evaluating the two-dimensional integral in (2.40) by numerical quadrature. The two-dimensional integration was done using the IMSL subroutine DTWODQ. This subroutine computes a two-dimensional integral using a Gauss–Kronrod rule. The integrals were evaluated to a relative precision of one percent.

If the sphere is very far from the wall, a positive value of U_s implies that the sphere is leading the fluid (for example, a negatively buoyant sphere in a downward-moving shear flow) and a negative value of U_s implies that the sphere lags the fluid (for example, a positively buoyant sphere in a downward-moving shear flow). When the distance between the wall and the sphere is large compared to the radius of the sphere, U_s is the velocity with which the sphere would sediment in an unbounded quiescent fluid.

A sphere which is suspended in a shear field will rotate. As a leading-order approximation it can be assumed that the angular velocity of a torque-free sphere in a linear shear flow field is given by Goldman, Cox & Brenner's (1967) expression. The effect of inertia of the angular velocity would result in a correction which is $O(Re)$. This $O(Re)$ correction in angular velocity would give rise to an $o(Re)$ contribution to the lift force which is formally of higher order than the lift force predicted by equation (2.38). Thus, the dimensionless lift force can be assumed to depend only on Λ_G, Λ_s and l/a. The cases for which the lift force has been studied are classified into those with a non-zero value of U_s and those for which U_s is zero.

3.1. *Non-zero values of U_s*

When $U_s \neq 0$, $V = U_s$, and the Reynolds number in (2.3) is defined by

$$Re = U_s a/\nu. \tag{3.1}$$

The condition that the wall lies in the inner region can be stated in terms of the Reynolds numbers as

$$Re_s \ll \kappa, \tag{3.2}$$

and

$$Re_G \ll \kappa^2. \tag{3.3}$$

The functions $\bar{U}_\rho$, $\bar{U}_\phi$ and $\bar{U}_z$ depend only on Λ_G and l/a for freely rotating and non-rotating spheres. Thus the integral, I, is a function of l/a and Λ_G. Positive values of Λ_G correspond to positive values of U_s and vice versa. Tables 1 and 2 contain the values of the integral, I, for l/a between 1.1 and 20.0 and Λ_G between -5.0 and 5.0 for a non-rotating sphere.

The case for which Λ_G is equal to zero in table 1 corresponds to a non-rotating sphere sedimenting in a stagnant fluid. Cox & Hsu's (1977) formula for the dimensionless lift force on a rigid sphere sedimenting near a flat vertical wall in a stagnant fluid is

$$I = 18\pi/32. \tag{3.4}$$

The value of I obtained by numerically evaluating the integral in (2.40) converges to within 1.0% of the value predicted by (3.4) for l/a greater than 6.

When the fluid is undergoing a shear flow, the value of Λ_G is non-zero. Cox & Hsu (1977) obtained an analytical expression for the lift force when the sphere is far from the wall. The expressions for the integral, I, for a sphere in a shear field according to Cox & Hsu's (1977) analysis are

$$I = \frac{18\pi}{32} - \frac{66\pi}{64}\Lambda_G\left(\frac{l}{a}\right) + \frac{366\pi}{576}\Lambda_G^2 \tag{3.5}$$

for a non-rotating sphere and

$$I = \frac{18\pi}{32} - \frac{66\pi}{64}\Lambda_G\left(\frac{l}{a}\right) + \frac{330\pi}{576}\Lambda_G^2 \tag{3.6}$$

for a freely rotating sphere. In figure 2, I has been plotted as a function of Λ_G for $l/a = 1.1$, 1.5, 5.0 and 20.0 for a non-rotating sphere. The values predicted by (3.5) are also shown in this figure. It can be seen from figure 2 that the percentage difference in the value of I obtained using (2.40) and that obtained using Cox & Hsu's expression decreases as l/a becomes large. Cox & Hsu's expression is a leading-order expression in κ as $\kappa \rightarrow 0$. When κ is $O(1)$, the near-wall effects alter the lift force as can be seen from figures 2 (*a*) and 2(*b*).

An interesting case for a non-rotating sphere is the one corresponding to $\Lambda_G = -1.0$. The value of I for this case for distances less than 3 sphere radii is shown in figure

l/a	$\Lambda_G = 0$	0.01	0.10	0.5	1.0
1.1	1.728	1.671	1.164	−0.706	−2.157
1.5	1.735	1.667	1.057	−1.255	−3.219
2.0	1.756	1.672	0.933	−1.940	−4.587
3.0	1.779	1.665	0.655	−3.410	−8.040
4.0	1.788	1.643	0.353	−4.961	−10.638
5.0	1.790	1.614	0.039	−6.537	−13.798
10.0	1.790	1.450	−1.567	−14.560	−29.864
20.0	1.790	1.119	−4.804	−30.729	−62.213
l/a	$\Lambda_G = 1.5$	2.0	3.0	4.0	5.0
1.1	−2.623	−2.104	1.888	9.8211	21.693
1.5	−4.166	−4.096	−0.896	6.379	17.632
2.0	−6.187	−6.739	−4.700	1.535	11.961
3.0	−10.595	−12.581	−13.343	−9.826	−2.030
4.0	−15.243	−18.777	−22.630	−22.196	−17.475
5.0	−19.991	−25.116	−32.161	−34.934	−33.436
10.0	−44.120	−57.333	−80.619	−99.723	−114.643
20.0	−92.670	−122.104	−177.89	−229.583	−277.169

TABLE 1. The integral, I, for a non-rotating sphere ($\Lambda_G > 0$).

l/a	$\Lambda_G = -0.01$	−0.1	−0.5	−1.0	−1.5
1.1	1.788	2.335	5.150	9.557	14.947
1.5	1.805	2.455	5.741	10.766	16.811
2.0	1.841	2.621	6.500	12.293	19.133
3.0	1.895	2.947	8.044	15.378	23.783
4.0	1.934	3.266	9.608	18.501	28.465
5.0	1.968	3.584	11.187	21.652	33.185
10.0	2.125	5.182	19.181	37.621	57.107
20.0	2.440	8.404	35.313	69.873	105.459

l/a	$\Lambda_G = -2.0$	−3.0	−4.0	−5.0
1.1	21.323	37.030	56.676	80.263
1.5	23.873	41.052	62.319	87.657
2.0	27.022	45.944	69.058	96.365
3.0	33.256	55.413	81.849	112.564
4.0	39.501	64.787	94.360	128.221
5.0	45.785	74.191	106.868	143.818
10.0	77.637	121.837	170.219	222.783
20.0	142.068	218.365	298.764	383.263

TABLE 2. The integral, I, for a non-rotating sphere ($\Lambda_G < 0$).

3. Since $V_{sph} = U_s + Ga(l/a)$, as $l/a \rightarrow 1$, $V_{sph} \rightarrow 0$ and $|Re_s| \rightarrow Re_G$. Leighton & Acrivos (1985) derived an expression for the lift force on a stationary sphere in a linear shear flow when the sphere touches the wall (i.e. $l/a = 1.0$). According to Leighton & Acrivos' analysis

$$I = 9.22. \tag{3.7}$$

From figure 3 it can be seen that the integral, I, tends to the Leighton–Acrivos limit as $l/a \rightarrow 1$. The lift force when l/a is exactly equal to 1 cannot be evaluated using a bi-polar coordinate system since the coordinate system becomes singular. Leighton & Acrivos derived (3.7) using a tangent plane coordinate system. The present analysis indicates

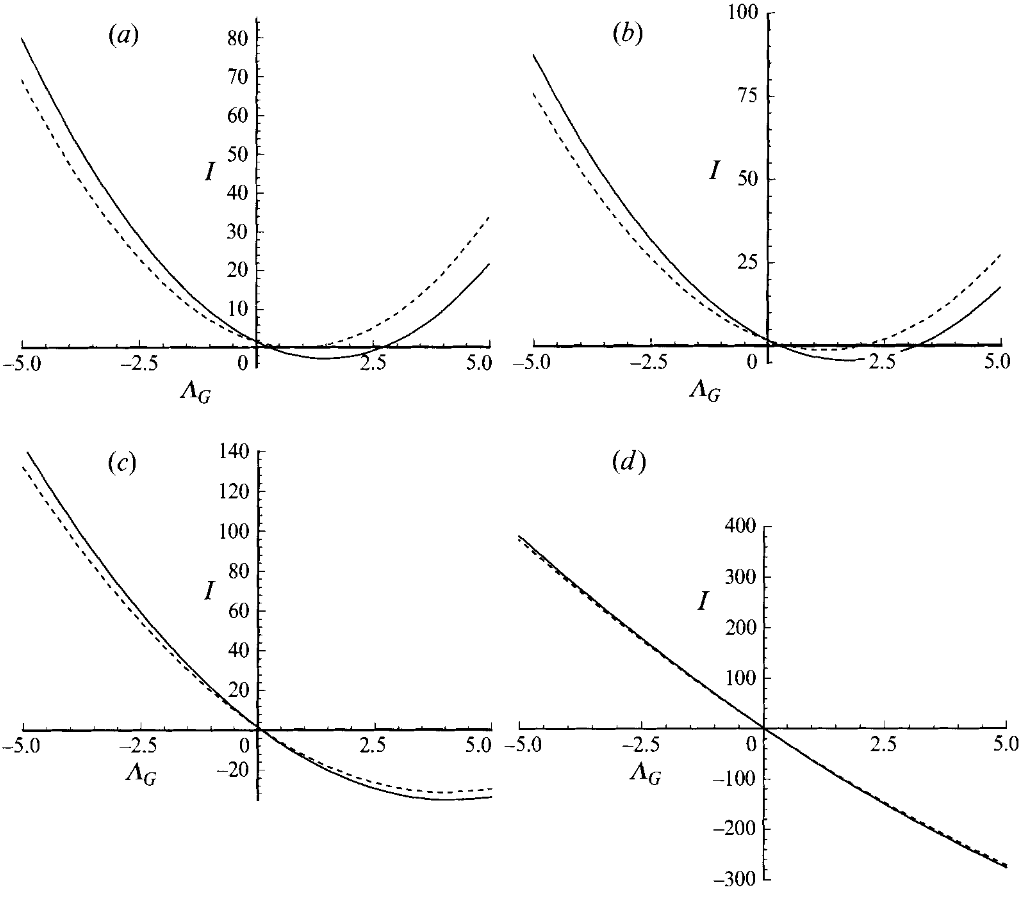

FIGURE 2. The integral, I, as a function of Λ_G for a non-rotating sphere when (a) $l/a = 1.1$; (b) 1.5, (c) 5.0, (d) 20.0. ----, (3.5); ——, (2.40).

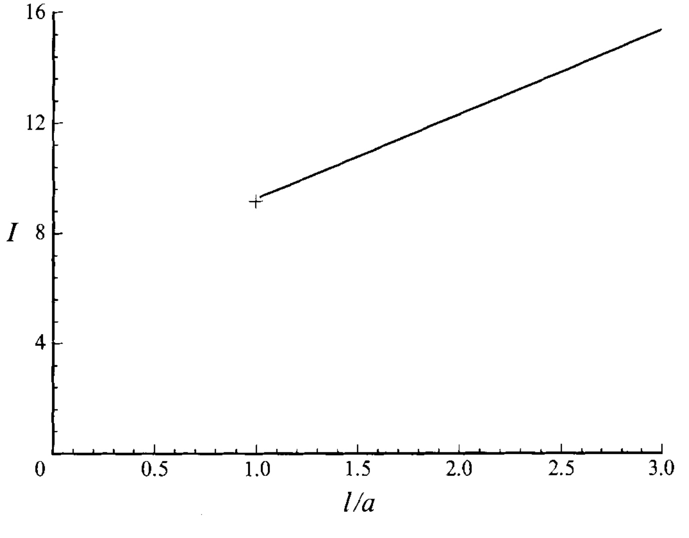

FIGURE 3. The integral, I, when $\Lambda_G = -1.0$ as a function of l/a: +, Leighton & Acrivos (1985); ——, (2.40).

l/a	$\Lambda_G = 0.01$	0.1	0.5	1.0	1.5
1.1	1.664	1.087	−1.127	−3.098	−4.184
1.5	1.656	0.965	−1.748	−4.312	−5.954
2.0	1.662	0.829	−2.499	−5.810	−8.175
3.0	1.654	0.537	−4.049	−8.918	−12.823
4.0	1.630	0.225	−5.639	−12.104	−17.606
5.0	1.600	−0.931	−7.245	−15.297	−22.439
10.0	1.435	−1.711	−15.329	−31.507	−46.745
20.0	1.104	−4.956	−31.530	−63.917	−95.381

l/a	$\Lambda_G = 2.0$	3.0	4.0	5.0
1.1	−4.386	−2.133	3.659	12.993
1.5	−6.677	−5.361	−0.365	8.314
2.0	−9.595	−9.600	−5.826	1.729
3.0	−15.767	−18.771	−17.923	−13.230
4.0	−22.146	−28.339	−30.678	−29.166
5.0	−28.098	−38.038	−43.643	−45.409
10.0	−61.043	−86.819	−108.844	−127.103
20.0	−125.925	−184.238	−238.871	−289.804

TABLE 3. The integral, I, for a rotating sphere ($\Lambda_G > 0$).

l/a	$\Lambda_G = -0.01$	−0.1	−0.5	−1.0	−1.5
1.1	1.796	2.407	5.471	10.098	15.610
1.5	1.814	2.542	5.974	11.464	17.708
2.0	1.850	2.721	6.957	13.103	20.193
3.0	1.906	3.061	8.071	16.325	25.040
4.0	1.946	3.389	10.178	19.531	29.846
5.0	1.981	3.712	11.785	22.739	34.651
10.0	2.139	5.323	19.843	38.838	58.774
20.0	2.455	8.052	36.011	71.166	107.242

l/a	$\Lambda_G = -2.0$	−3.0	−4.0	−5.0
1.1	22.007	37.803	56.447	78.977
1.5	24.874	41.965	62.737	87.191
2.0	28.230	47.137	69.825	96.293
3.0	34.718	56.958	83.046	112.982
4.0	41.124	66.569	95.863	129.007
5.0	48.023	76.147	108.098	144.890
10.0	79.648	124.215	172.542	224.625
20.0	144.242	221.008	301.463	385.610

TABLE 4. The integral, I, for a rotating sphere ($\Lambda_G < 0$).

that Leighton & Acrivos' expression should give a fairly accurate value for the lift force even when the sphere does not touch the wall provided l/a is close to unity.

The values of the lift force for a freely rotating sphere as a function of Λ_G and l/a when U_s is non-zero are given in tables 3 and 4. As mentioned in §3, Goldman *et al.*'s (1967) expression was used to estimate the angular velocity of the sphere. The values of I for a freely rotating sphere for $l/a = 1.1$ and $l/a = 5.0$ are plotted as a function of Λ_G in figure 4. The effect of rotation is very small and becomes important only when the shear is large and the sphere is close to the wall.

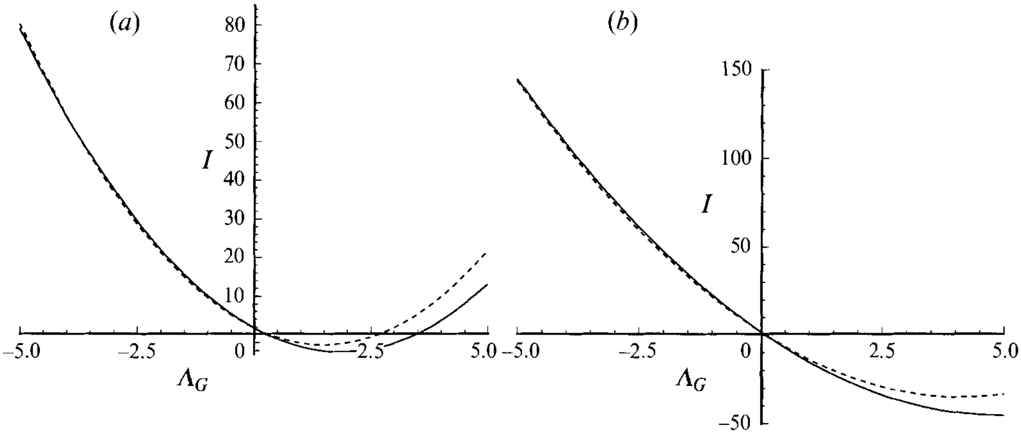

FIGURE 4. The integral, I as a function of Λ_G for (a) $l/a = 1.1$, and (b) $l/a = 5.0$, rotating and non-rotating spheres: ---, (2.40) for a non-rotating sphere; ——, (2.40) for a rotating sphere.

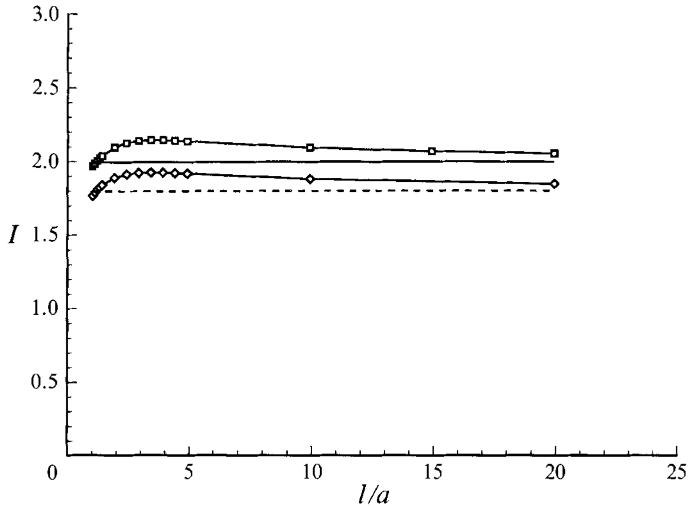

FIGURE 5. The integral, I, as a function of l/a when $U_s = 0$: ---, (3.10); ——, (3.9); □, (2.40) for a non-rotating sphere; ◇, (2.40) for a rotating sphere.

3.2. *Lift force when* $U_s = 0$

This case corresponds to a neutrally buoyant sphere when $l/a \to \infty$. The characteristic velocity is Ga and the Reynolds number defined in (2.3) is

$$Re = Ga^2/\nu. \tag{3.8}$$

The integral, I, as a function of l/a for rotating and non-rotating spheres as a function of l/a is shown in figure 5. The value of I varies very little with l/a and the lift force points away from the wall.The effect of rotation is to decrease the lift force. According to Cox & Hsu's expression, the integral, I, for a neutrally buoyant sphere is given by

$$I = \frac{366\pi}{576}, \tag{3.9}$$

for a non-rotating sphere and

$$I = \frac{330\pi}{576} \tag{3.10}$$

for a freely rotating sphere. The lift force predicted by (2.38) converges rapidly to the value predicted by (3.9) and (3.10).

4. A simple equation for the lift force

To evaluate the lift force on a sphere near a wall, the integral in (2.38) has to be evaluated numerically. This can be computationally expensive in particle trajectory calculations especially when there are a large number of particles very close to the wall. Hence it is useful to obtain a simple equation for the lift force. The functions $\bar{U}_\rho$, $\bar{U}_\phi$ and $\bar{U}_z$ depend linearly on Λ_G. Hence the integral, I, defined by (2.40) can be expressed as a quadratic function of Λ_G. The coefficients of this second-degree polynomial in Λ_G can be obtained easily by the least-squares method. The coefficients obtained in this manner are exact since the lift integral has an exact quadratic dependence on Λ_G. The coefficients of Λ_G^0, Λ_G^1 and Λ_G^2 depend on κ. These coefficients can be expressed as a function of κ and the unknown coefficients of the powers of κ were computed by nonlinear minimization and the following equation was obtained for a non-rotating sphere:

$$\begin{aligned} I = {} & [1.7716+0.2160\kappa-0.7292\kappa^2+0.4854\kappa^3] \\ & -\left[\frac{3.2397}{\kappa}+1.1450+2.0840\kappa-0.9059\kappa^2\right]\Lambda_G \\ & +[2.0069+1.0575\kappa-2.4007\kappa^2+1.3174\kappa^3]\Lambda_G^2. \end{aligned} \tag{4.1}$$

Equation (4.1) fits the data in tables 1 and 2 very accurately. For example, in the Leighton–Acrivos limit, the value of I predicted by (4.1) is 9.28. In a similar manner, it was found that the equation

$$\begin{aligned} I = {} & [1.7631+0.3561\kappa-1.1837\kappa^2+0.845163\kappa^3] \\ & -\left[\frac{3.24139}{\kappa}+2.6760+0.8248\kappa-0.4616\kappa^2\right]\Lambda_G \\ & +[1.8081+0.879585\kappa-1.9009\kappa^2+0.98149\kappa^3]\Lambda_G^2 \end{aligned} \tag{4.2}$$

fits the data in tables 3 and 4 for a freely rotating sphere very well. Equations (4.1) and (4.2) can be used to evaluate the lift force without actually computing the integral, I, numerically.

5. Conclusion

Equation (2.38) predicts the lift force to $O(Re)$ when $Re \ll 1$ and accounts for the finite size of the sphere. It can be used for any distance, l, between the wall and the sphere provided $l \ll \min(\nu/U_s, (\nu/G)^{\frac{1}{2}})$. Cox & Brenner (1968) also used the technique mentioned in §2 to express the lift force as a volume integral involving two creeping flow solutions. In the present analysis, the exact creeping flow solutions have been used while evaluating the volume integral. The integrand in (2.24) is $v_i^{(0)}(u_j^{(0)}\,\partial u_i^{(0)}/\partial x_j)$. Cox & Brenner (1968) neglected the contributions to $u_j^{(0)}\,\partial u_i^{(0)}/\partial x_j$ that result in terms which are $O(\kappa)$ in the expression for the lift force. Such terms are insignificant when the distance between the sphere and the wall is very large compared to its radius. The volume integral was evaluated over the entire space occupied by the fluid while neglecting the finite size of the sphere. The disturbance velocity was approximated as that due to a point force singularity whose strength is equal to the Stokes drag and a force dipole singularity. This gives an expression which is valid to $O(\kappa^{-1})$ for a non-neutrally buoyant sphere and to $O(\kappa^0)$ for a neutrally buoyant sphere when κ is very small compared to unity. It should be noted that this expression does not contain all the terms that are at least $O(\kappa^0)$. Thus, even though the percentage difference between the value of I obtained by numerically evaluating the integral (2.40) and that obtained using (3.5) and (3.6) decreases as $\kappa \rightarrow 0$, the absolute difference does not change considerably. The difference in the lift force predicted by (2.38) and (3.5) and (3.6) increases as κ becomes $O(1)$. The difference in the lift predicted by (2.38) and (3.5) and (3.6) is due to terms which are of higher order in κ than those which have been retained in Cox & Brenner's (1968) and Cox & Hsu's (1977) expressions. The value of the lift force obtained by numerically evaluating the integral in (2.24) indicates that these higher-order terms are important when the sphere is very close to the wall.

Lovalenti has derived an expression for the lift force for small κ and arbitrary angular speed by modifying the results derived by Cox & Brenner (1968) and Cox & Hsu (1977) (see the Appendix by Lovalenti). The disturbance flow was approximated as that due to a point force singularity, whose strength is the Stokes drag multiplied by a factor which takes into account the $O(\kappa)$ correction to the Stokes drag due to the presence of the wall, and a force dipole. Instead of evaluating the integral over the entire space occupied by the fluid, the finite size of the sphere was also taken into consideration. This results in an expression which is valid when $\kappa \ll 1$ and contains all the terms which are at least $O(\kappa^0)$ and has the exact $O(1)$ asymptotic form as $\kappa \rightarrow 0$. When expressed in terms of the integral, I, defined by (2.40), Lovalenti's expression becomes

$$I = \frac{18\pi}{32} - \frac{66\pi}{64}\left[\frac{1}{\kappa} + \frac{374\pi}{1056}\right]\Lambda_G + \frac{366\pi}{576}\Lambda_G^2 + O(\kappa) \tag{5.1}$$

for a non-rotating sphere. The value of I obtained using (5.1) agrees very well with the data in tables 1 and 2 when the sphere is several radii from the wall. This can be found by comparing the leading-order behaviour of (4.1) as $\kappa \rightarrow 0$. The coefficients of the terms which are at least $O(1)$ as $\kappa \rightarrow 0$ in (4.1) agree very well with the coefficients of similar terms in (5.1).

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Appendix A. The lift force on a sphere in a simple shear flow near a plane wall

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Using existing results for a sphere translating near a flat wall in a shear flow at low Reynolds number, we shall derive the lift force on a sphere of radius a when its distance from the wall l satisfies

$$a \ll l \ll \min(\nu/U_s, (\nu/G)^{\frac{1}{2}}), \tag{A 1}$$

where ν is the kinematic viscosity of the fluid, U_s is the magnitude of the sphere's slip velocity relative to the fluid, and G is the shear rate of the imposed flow. The lower bound in the above inequality allows the sphere's weak interaction with the wall to be adequately accounted for by a point-force plus force-dipole description of the sphere. The upper bound provides for the condition that the wall is in the inner region of expansion and allows for the use of regular perturbation techniques. Our goal is to collect all terms up to $O(1)$ in inverse powers of the particle distance from the wall l while neglecting those of higher order since they decay as the sphere moves farther from the wall.

For a steady, planar, simple shear flow in the 2-direction which increases in the 1-direction, the disturbance flow $\boldsymbol{u}'$ created by the particle translating with velocity U_s in the 2-direction relative to the shear flow evaluated at the sphere centre is described by the Navier–Stokes equations as

$$-\nabla p + \mu \nabla^2 \boldsymbol{u}' = \rho\left(-U_s \frac{\partial \boldsymbol{u}'}{\partial r_2} + \boldsymbol{u}' \cdot \nabla \boldsymbol{u}' + G u_1' \boldsymbol{e}_2 + G r_1 \frac{\partial \boldsymbol{u}'}{\partial r_2}\right) \tag{A 2}$$

$$\equiv \boldsymbol{f}(\boldsymbol{u}'), \tag{A 3}$$

$$\nabla \cdot \boldsymbol{u}' = 0, \tag{A 4}$$

where μ and ρ are the viscosity and density of the fluid, and the coordinate system has its origin at the centre of the sphere. Here, the wall surface is at $r_1 = -l$ on which the fluid satisfies the no-slip boundary condition.

Now since a reversibility argument demonstrates that the Stokes equations result in no lift force perpendicular to the wall, the existence of a lift force must be due solely to inertial effects. Then the general reciprocal theorem provides the following expression for the lift force F_l in the 1-direction:

$$F_l = \int_{V_f} \boldsymbol{f}(\boldsymbol{u}') \cdot \boldsymbol{v}_1 \, \mathrm{d}V, \tag{A 5}$$

where V_f represents the entire volume of fluid surrounding the sphere and bounded by the wall. Here, $\boldsymbol{v}_1$ is the Stokes velocity field produced by the sphere, a distance l from the wall, translating with unit velocity in the 1-direction in a quiescent fluid (in the absence of a shear field).

Owing to condition (A 1), we can approximate F_l by replacing $\boldsymbol{u}'$ with the corresponding Stokes solution for the disturbance flow, $\boldsymbol{v}$, as a regular perturbation approach:

$$F_l \sim \int_{V_f} \boldsymbol{f}(\boldsymbol{v}) \cdot \boldsymbol{v}_1 \, \mathrm{d}V. \tag{A 6}$$

The errors in the above expression are of higher order in Reynolds number as can be seen from both a regular and singular perturbation analysis. The integral will remain convergent because far from the particle–wall system the disturbance appears as that due to a dipole and thus decays as $O(r^{-2})$ making the integrand $O(r^{-4})$.

The contributions to the lift force can be divided into two sources: that due to the presence of the wall and that due to the finite size of the sphere that would exist in the absence of the wall. The former lift force contribution was evaluated by Cox & Hsu (1977), while the latter was evaluated by Saffman (1965). The results from Cox & Hsu were obtained by using a point-force plus force-dipole description of the sphere's disturbance velocity, a representation which is valid under the assumption that the sphere is far from the wall. In their analysis the nonlinear term of f, $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$, is neglected since it produces a contribution to the lift which can be shown to decay as $O(l^{-1})$. Thus, f can be treated as being linear in $\boldsymbol{v}$ and the point force and force dipole can be accounted for separately. If the point-forced velocity field is used in (A 6) for both $\boldsymbol{v}$ and $\boldsymbol{v}_1$, two terms are obtained. From the work of Cox & Hsu, the first is due to the first term on the right-hand side of (A 2):

$$\frac{18\pi}{32}\mu a U_s\left(\frac{aU_s}{\nu}\right), \tag{A 7}$$

and the second from the last two terms of (A 2):

$$-\frac{66\pi}{64}\mu a U_s\left(\frac{alG}{\nu}\right). \tag{A 8}$$

It is important to note that these two terms were computed using a point force of magnitude equal to the Stokes drag on a sphere in an unbounded domain, $-6\pi\mu a U_s$. The term given by (A 8) must be corrected to ensure that we obtain all terms which do not decay for large l. This is accomplished by including a modification of the point force due to the presence of the wall. For motion parallel to the wall the magnitude of the point force should have a multiplicative factor of $(1+9a/16l)$, while for motion perpendicular to the wall it should have a factor of $(1+9a/8l)$ (see Happel & Brenner 1965). If these factors are used in (A 6), the corrected term of (A 8) becomes

$$-\frac{66\pi}{64}\mu a U_s\left(\frac{alG}{\nu}\right)\left(1+\frac{27}{16}\frac{a}{l}\right). \tag{A 9}$$

When the force-dipole description of the sphere is used for $\boldsymbol{v}$ in the last two terms of (A 2), while the point-force description is used for $\boldsymbol{v}_1$, Cox & Hsu found contributions to the lift force from (A 6) given by

$$\frac{6\pi(61)}{144\times 4}\mu a^2 G\left(\frac{a^2G}{\nu}\right), \tag{A 10}$$

for a sphere prevented from rotating, and

$$\frac{6\pi(55)}{144\times 4}\mu a^2 G\left(\frac{a^2G}{\nu}\right), \tag{A 11}$$

for a sphere free to rotate.

In evaluating the above results ((A 7), (A 8), (A 10), and (A 11)), Cox & Hsu performed the integration in (A 6) by extending the volume of integration to the entire volume of space, ignoring the finite size of the sphere. The error made in doing this yields contributions to the lift force which do not decay for large l. These contributions may be evaluated by neglecting the presence of the wall and using the disturbance Stokes flow fields for the motion of the sphere in an unbounded domain. This is carried out by first evaluating the integral (A 6) over an unbounded fluid domain *outside* the sphere with the velocity fields replaced by those for the sphere motion in an unbounded domain (these fields are well known), while taking care not to include the point-force or force-dipole contributions from the above four terms already evaluated by Cox & Hsu. Then, in order to correct the error in the Cox & Hsu analysis, one must subtract the integral over the volume of the *sphere* of these excluded point-force and force-dipole contributions. The result yields the second-order Saffman lift force (Saffman 1965), which is determined from a consideration of the inner expansion problem:

$$\pi\mu a U_s\left(\frac{11}{8}\frac{a^2G}{\nu}-\frac{a^2\Omega}{\nu}\right), \tag{A 12}$$

where Ω is the magnitude of the angular velocity of the sphere in the 3-direction.

If we now combine all these contributions we can obtain an expression for the lift force to leading order in Reynolds number and appropriate when $a/l \ll 1$:

$$F_l = \frac{18\pi}{32}\mu a U_s\left(\frac{aU_s}{\nu}\right)-\frac{66\pi}{64}\mu a U_s\left(\frac{alG}{\nu}\right)\left(1+\frac{27}{16}\frac{a}{l}\right)$$
$$+\frac{6\pi(61)}{144\times 4}\mu a^2 G\left(\frac{a^2G}{\nu}\right)+\tfrac{11}{8}\pi\mu a U_s\left(\frac{a^2G}{\nu}\right)+O(a/l), \tag{A 13}$$

for a sphere prevented from rotating, and for a sphere free to rotate:

$$F_l = \frac{18\pi}{32}\mu a U_s\left(\frac{aU_s}{\nu}\right)-\frac{66\pi}{64}\mu a U_s\left(\frac{alG}{\nu}\right)\left(1+\frac{27}{16}\frac{a}{l}\right)$$
$$+\frac{6\pi(55)}{144\times 4}\mu a^2 G\left(\frac{a^2G}{\nu}\right)+\tfrac{7}{8}\pi\mu a U_s\left(\frac{a^2G}{\nu}\right)+O(a/l), \tag{A 14}$$

where we have set $\Omega = G/2$ for a freely rotating sphere.† As an added note, the expression for a reversal in the direction of the shear flow or the slip velocity can be obtained by simply changing the sign of G or U_s in the above two expressions. Also, the corresponding expression for an arbitrary sphere rotation speed can be found by a linear interpolation of (A 10) and (A 11) since this contribution scales linearly with the dipole and the dipole varies linearly with the angular speed of the sphere. Thus, for arbitrary rotation speed the lift force is given by

$$F_l = \frac{18\pi}{32}\mu a U_s\left(\frac{aU_s}{\nu}\right)-\frac{66\pi}{64}\mu a U_s\left(\frac{alG}{\nu}\right)\left(1+\frac{27}{16}\frac{a}{l}\right)$$
$$+\frac{6\pi(61)}{144\times 4}\mu a^2 G\left(\frac{a^2G}{\nu}\right)\left(1-\frac{6}{61}\frac{2\Omega}{G}\right)$$
$$+\tfrac{11}{8}\pi\mu a U_s\left(\frac{a^2G}{\nu}\right)\left(1-\frac{4}{11}\frac{2\Omega}{G}\right)+O(a/l). \tag{A 15}$$

† This value of the rotation speed is correct for zero Reynolds number in the absence of any bounding walls. The corrections for finite Reynolds number or for walls will bring only a smaller-order correction to the lift force than those already provided.

REFERENCES

CHERUKAT, P., MCLAUGHLIN, J. B. & GRAHAM, A. L. 1994 The inertial lift on a rigid sphere in a linear shear flow field. *Intl J. Multiphase Flow* (accepted for publication).

COX, R. G. 1963 The steady motion of a particle of arbitrary shape at small Reynolds numbers. *J. Fluid Mech.* **23**, 625–643.

COX, R. G. & BRENNER, H. 1968 The lateral migration of solid particles in Poiseuille flow: I. Theory. *Chem. Engng Sci.* **23**, 147–173.

COX, R. G. & HSU, S. K. 1977 The lateral migration of solid particles in a laminar flow near a plane. *Intl J. Multiphase Flow* **3**, 201–222.

DREW, D. A. 1988 The lift force on a sphere in the presence of a wall. *Chem. Engng Sci.* **43**, 769–773.

GOLDMAN, A. J., COX, R. G. & BRENNER, H. 1967 Slow viscous motion of a sphere parallel to a plane wall-II: Couette flow. *Chem. Engng Sci.* **22**, 653–660.

HAPPEL, J. & BRENNER, H. 1965 *Low Reynolds Number Hydrodynamics*. Prentice-Hall.

HO, B. P. & LEAL, L. G. 1974 Inertial migration of rigid spheres in two-dimensional uni-directional flows. *J. Fluid Mech.* **65**, 365–400.

JOHANSSON, H., OLGRAD, G. & JERQVIST, A. 1970 Radial particle migration in plug flow – a method for solid-liquid separation and fractionation. *Chem. Engng Sci.* **25**, 365–372.

LEIGHTON, L. A. & ACRIVOS, A. 1985 The lift on a small sphere touching a plane in the presence of a simple shear flow. *Z. Agnew. Math. Phys.* **36**, 174–178.

LIN, C. J., LEE, K. J. & SATHER, N. F. 1970 Slow motion of two spheres in a shear field. *J. Fluid Mech.* **43**, 35–47.

MCLAUGHLIN, J. B. 1989 Aerosol particle deposition in numerically simulated channel flow. *Phys. Fluids* A **1**, 1211–1224.

MCLAUGHLIN, J. B. 1991 Inertial migration of a small sphere in linear shear flows. *J. Fluid Mech.* **224**, 261–274.

MCLAUGHLIN, J. B. 1993 The lift on a small sphere in wall-bounded linear shear flows. *J. Fluid Mech.* **246**, 249–265.

POZRIKIDIS, C. 1992 *Boundary Integral and Singularity Methods for Linearized Viscous Flow*. Cambridge University Press.

PROUDMAN, I. & PEARSON, J. R. A. 1957 Expansions at small Reynolds numbers for the flow past a sphere and a circular cylinder. *J. Fluid Mech.* **2**, 237–262.

RUBINOW, S. I. & KELLER, J. B. 1961 The transverse force on a spinning sphere moving in a viscous fluid. *J. Fluid Mech.* **22**, 385–400.

SAFFMAN, P. G. 1965 The lift on a small sphere in a slow shear flow. *J. Fluid Mech.* **22**, 385–400.

SCHONBERG, J. A. & HINCH, E. J. 1989 The inertial migration of a sphere in Poiseuille flow. *J. Fluid Mech.* **203**, 517–524.

VASSEUR, P. & COX, R. G. 1976 The lateral migration of a spherical particle in two-dimensional shear flows. *J. Fluid Mech.* **78**, 385–413.